1. Model of the process

Modern methods of design of advanced controllers usually require high quality models of the process. A classical procedure of a model development consists of the following steps:

- development of the mathematical model based on physics of the process,
- simplification of the model and its transformation into a standard form,
- development of a simulation model,
- tuning of the model parameters (identification),
- verification of the model.

In the next sections we will execute the steps given in the above list for the PT-326 process.

1.1 Assumptions and notations

The model of PT 323 process (Fig.2.1) can be divided into the following submodels:

- model of heat transfer from the heater to air,
- model of the heater,
- heat transportation model along the pipe.

The models described in following sections are based on the following assumptions:

- flow of the air in the pipe is incompressible: density of the air may be considered to be constant along the pipe,
- flow in the pipe is steady: the speed is constant, the mass of air per time unit crossing any surface normal to the flow direction is a constant value,

- mixing of the air is only radial. This condition is well met, since the radius of the pipe is much smaller than its length and the air flow speed is large. Therefore one can say that **heat transfer along the pipe** has purely **transportation character,**
- heat exchange through the walls of the pipe can be neglected,
- heating process is linear almost everywhere in the range of the heating power (this assumption can be checked experimentally),
- dynamics of the temperature sensor can be neglected due to its small heat capacity (relatively to the heater).

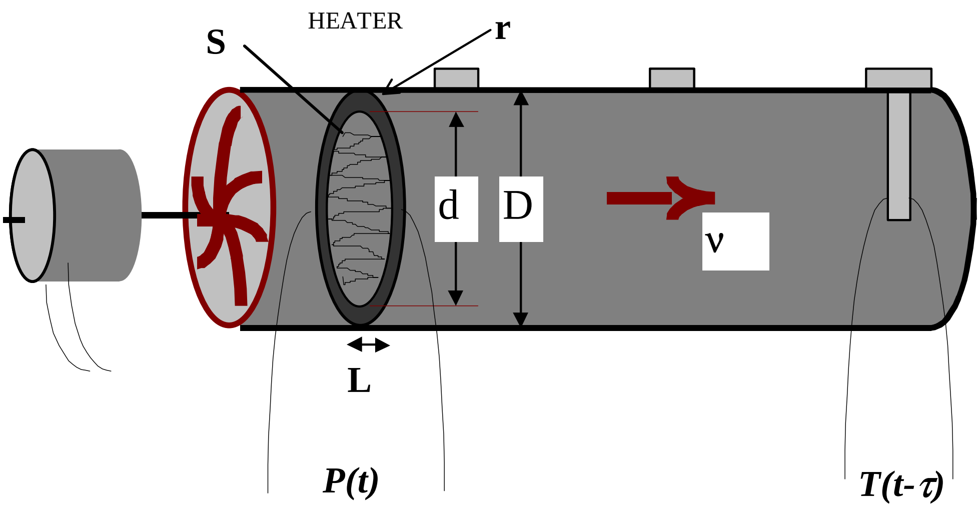

Fig. 2.1 PT 326 system

The following notation is introduced for the models of the PT 326 system (see Fig.2.1):

t - time (variable) [s],

L - length of the heat exchange area [m],

S - cross sectional area of the pipe at the narrow (heating point) [m^2],

l - space variable of heat exchange [m], $l \in [0, L]$,

$H(t,l)$ - temperature of the heater measured relatively to the room temperature, the function of time and space [$°C$],

$T(t,l)$ - temperature of the air flowing through the heater measured relatively to the room temperature, the function of time and space [$°C$],

v - speed of the air flow [$\frac{m}{s}$],

$P(t)$ - power supplied to the heater [W],

r - heat exchange circumference [m],

K - heat exchange coefficient [$\frac{J}{s \cdot m^2 \cdot °\mathrm{C}}$],

m - mass of the heater [kg],

c_g - heat capacity of the heater $[\frac{J}{kg \cdot {}^\circ C}]$,

c - heat capacity of the air $[\frac{J}{kg \cdot {}^\circ C}]$,

ρ - air density at constant temperature (at 40°C- it is the middle of the temperature range) $[\frac{kg}{m^3}]$,

Some of physical constants are known or can be easily measured or calculated:

$c_g = 385$	$c = 718 \div 1005$ [*)]	$r \approx 1.0$	$P_{max} = 70$
$\rho = 1.128$	$L \approx 0.002$	$S = \frac{\pi d^2}{4} = 0.001134$	

[*)] the upper limit given for an isobaric heating process, the lower limit for an isochoric process.

1.2 Model of the heater

Consider an infinitesimal, uniaxial element *dl* of the heater. Its mass is:

$$dm = \frac{dl}{L} \cdot m$$

The internal energy of this element can be calculated as:

$$dQ(t,l) = dm \cdot c_g \cdot H(t,l) = \frac{dl}{L} \cdot m \cdot c_g \cdot H(t,l).$$

A stream of heat transferred into the considered element of the heater volume can be expressed as:

$$F_{we}(t,l) = dP(t) = \frac{dl}{L} \cdot P(t).$$

Stream of heat flowing outside the considered heat element can be calculated as:

$$F_{wy}(t,l) = K \cdot r \cdot dl \cdot \left(H(t,l) - T(t,l)\right).$$

The general equation describing the dynamics of the heat flow gives:

$$\frac{dQ(t,l)}{dt} = F_{we}(t,l) - F_{wy}(t,l),$$

$$\frac{\partial(\frac{dl}{L} \cdot m \cdot c_g \cdot H(t,l))}{\partial t} = \frac{dl}{L} \cdot P(t) - K \cdot r \cdot dl \cdot \left(H(t,l) - T(t,l)\right)$$

As the parameters m, dl and c_g are constant, we obtain:

$$\frac{dl}{L}\cdot m\cdot c_g\cdot\frac{\partial H(t,l)}{\partial t}=\frac{dl}{L}\cdot P(t)-K\cdot r\cdot dl\cdot\left(H(t,l)-T(t,l)\right),$$

and finally:

$$\boxed{\frac{\partial H(t,l)}{\partial t}=\frac{1}{m\cdot c_g}\cdot P(t)-\frac{K\cdot r\cdot L}{m\cdot c_g}\cdot\left(H(t,l)-T(t,l)\right)} \qquad (2.1)$$

where the initial condition is: $H(0,l)=0$ (at t=0 temperature of the heater is equal to the ambient temperature).

1.3 Heat exchange model

In order to develop the model of heat transfer from the heater to air consider an imaginary volume of air moving in the tube, which passes through the surface *S* at a certain moment *t* with the speed *v*. At the time *dt* later the volume will pass an infinitesimal distance *dl*, as shown in Fig. 3.1. The considered mass of the air volume can be calculated as:

$$dm=\rho\cdot S\cdot dl\,.$$

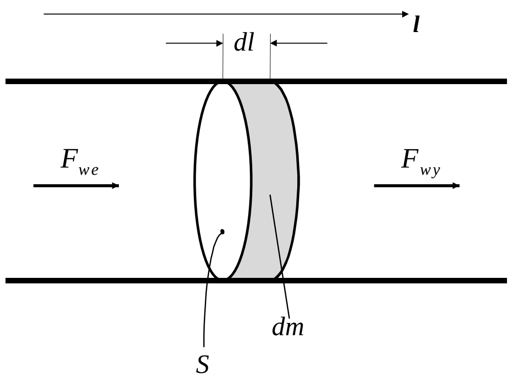

Fig. 2.2 Basic heat exchange model

The internal energy *dQ* of the air in the volume *dV* can be calculated as:

$$dQ(t,l)=dm\cdot c\cdot T(t,l)\;=\;S\cdot\rho\cdot dl\cdot c\cdot T(t,l)\,.$$

The stream of heat transferred into the considered air volume is expressed as:

$$F_{we}(t,l)=S\cdot\rho\cdot v\cdot c\cdot T(t,l-dl)+K\cdot r\cdot dl\cdot\left(H(t,l)-T(t,l)\right),$$

where : $S\cdot\rho\cdot v$ - is the stream of mass flowing into the considered volume,

$T(t,l-dl)$ - is the temperature of air at the previous infinitesimal distance *dl*.

The heat transferred outside the considered air volume is expressed as:

$$F_{wy}(t,l)=S\cdot\rho\cdot v\cdot c\cdot T(t,l),$$

where:

$T(t,l)$ - is the temperature of the air inside the considered volume.

According to the assumptions that there are no any other sources and sinks of energy in the considered region, the general equation describing the dynamics of the heat flow can be written in the form:

$$\frac{\partial(\rho \cdot S \cdot dl \cdot c \cdot T(t,l))}{\partial t} = S \cdot \rho \cdot v \cdot c \cdot T(t,l-dl) + K \cdot r \cdot dl \cdot \left(H(t,l) - T(t,l)\right) + \\ - S \cdot \rho \cdot v \cdot c \cdot T(t,l)$$

or finally:

$$\rho \cdot S \cdot dl \cdot c \cdot \frac{\partial(T(t,l))}{\partial t} = S \cdot \rho \cdot v \cdot c \cdot \left(T(t,l-dl) - T(t,l)\right) + K \cdot r \cdot dl \cdot \left(H(t,l) - T(t,l)\right)$$

$$\frac{\partial(T(t,l))}{\partial t} = -v \cdot \frac{T(t,l) - T(t,l-dl)}{dl} + \frac{K \cdot r}{\rho \cdot c \cdot S} \cdot \left(H(t,l) - T(t,l)\right),$$

For $dl \to 0$ we have:

$$\boxed{\frac{\partial(T(t,l))}{\partial t} = -v \frac{\partial(T(t,l))}{\partial l} + \frac{K \cdot r}{\rho \cdot c \cdot S} \cdot \left(H(t,l) - T(t,l)\right)} \qquad (2.2)$$

Equation (2.2) together with the initial and boundary conditions $T(0,l)=0$, $T(t,0)=0$ forms the heat exchange model.

1.4 Distributed parameter model of PT 326 process

The mathematical techniques being used in this chapter are based upon the application of two-dimensional Laplace transformation for the equations (2.1) and (2.2). We apply the transformation with respect to the time variable (Laplace operator *s*) and with respect to the space variable (Laplace operator *p*), assuming the initial and boundary conditions equal to zero (for a simplicity, the same notations are used for original and transformed variables):

$$s \cdot H(s,p) = \frac{1}{m \cdot c_g p} \cdot P(s) - \frac{K \cdot r \cdot L}{m \cdot c_g} \cdot H(s,p) + \frac{K \cdot r \cdot L}{m \cdot c_g} \cdot T(s,p), \qquad (2.3)$$

$$s \cdot T(s,p) = -v \cdot p \cdot T(s,p) + \frac{K \cdot r}{\rho \cdot c \cdot S} \cdot H(s,p) - \frac{K \cdot r}{\rho \cdot c \cdot S} \cdot T(s,p). \qquad (2.4)$$

We can determine $H(s,p)$ from equation (2.3) and substitute the result into the equation (2.4) obtaining a direct relation between $P(s)$ and $T(s,p)$. Finally, we can apply the inverse transform with respect to the space variable to find:

$$T(s,l) = \frac{\beta}{m \cdot c_g \cdot s \cdot (s + \alpha + \beta)} \cdot \left(1 - e^{-\frac{s(s+\alpha+\beta)}{v(\alpha+s)} \cdot l}\right) \cdot P(s),$$

where:

$$\alpha = \frac{K \cdot r \cdot L}{m \cdot c_g}, \; \beta = \frac{K \cdot r}{\rho \cdot c \cdot S}.$$

The transfer function of the PT 326 process calculated for the end of the heater (*l = L)* becomes:

$$G_{WC}(s,L) = \frac{T(s,L)}{P(s)} = \frac{\beta}{m \cdot c_g \cdot s \cdot (s+\alpha+\beta)} \cdot \left(1 - e^{-\frac{s(s+\alpha+\beta)}{v(\alpha+s)} \cdot L}\right).$$

Including into the model the heat transportation delay between the heater and sensor position we have:

$$\boxed{G(s) = \frac{T(s)}{P(s)} = \frac{\beta}{m \cdot c_g \cdot s \cdot (s+\alpha+\beta)} \left(1 - e^{-\frac{s(s+\alpha+\beta)}{v(\alpha+s)} \cdot L}\right) \cdot e^{-s\cdot\tau}} \qquad (2.5)$$

where: $e^{-s\cdot\tau}$ is the term modelling heat transportation time delay τ, $T(s)$ is the air temperature in the sensor location and *P(s)* is the power of the heater (the control variable).

1.5 Simplified model

In order to effectively apply the model (2.5) for the controller design it is necessary to replace it by a lumped approximation. For this purpose we replace the expression in (2.5):

$$\left(1 - e^{-\frac{s(s+\alpha+\beta)}{v(\alpha+s)} \cdot L}\right) = \left(\frac{e^{\frac{s(s+\alpha+\beta)}{v(\alpha+s)} \cdot L} - 1}{e^{\frac{s(s+\alpha+\beta)}{v(\alpha+s)} \cdot L}}\right)$$

by its Taylor series leaving only the first two terms:

$$e^{\frac{s(s+\alpha+\beta)}{v(\alpha+s)} \cdot L} = 1 + \frac{L}{v} \cdot \frac{\alpha+\beta}{\alpha} \cdot s.$$

We obtain the following lumped parameter model of the PT 326 system:

$$G(s) = \frac{\beta}{m \cdot c_g \cdot (s+\alpha+\beta)\left(s + \frac{v \cdot \alpha}{L(\alpha+\beta)}\right)} \cdot e^{-s\cdot\tau}$$

Introducing the following notations:

$K1 = K\,r\,[\frac{J}{m \cdot s \cdot {}^\circ\mathrm{C}}]$ - modified heat exchange coefficient between heater and air,

$K2 = m\,c_g\,[\frac{J}{{}^\circ\mathrm{C}}]$ - modified heater constant,

$K3 = \rho\,c\,S\,[\frac{J}{m \cdot {}^\circ\mathrm{C}}]$ - modified air constant,

we can introduce the **simplified model of PT 326** in the form:

$$G(s)=\frac{\frac{K1}{K2\cdot K3}}{\left(s+\frac{K1\cdot L}{K2}+\frac{K1}{K3}\right)\left(s+\frac{v}{L+\frac{K2}{K3}}\right)}\cdot e^{-s\cdot\tau} \qquad (2.\ 6)$$