Twin Rotor Aerodynamical System

(draft version)

1. Model and parameters

Modern methods of design and adaptation of real time controllers require high quality mathematical models of the system. For high order, non-linear cross-coupled systems classical modelling methods (based on Lagrange equations) are often very complicated. That is why we decided to use a simpler approach, which is based on block diagram representation of the system, very suitable for SIMULINK environment.
A block diagram of the TRAS model is shown in Fig. 1.1. The control voltages U_h and U_v are inputs to the DC-motors which drive the rotors (PWM mode).
Rotation of a propeller produces an angular momentum which, according to the law of conservation of angular momentum, must be compensated by the remaining body of the TRAS beam. This results in the interaction between two transfer functions, represented by the moment of inertia of the motors with propellers k_{hv} and k_{vh} in Fig. 1.1. This interaction directly influences the velocities of the beam in both planes. The forces F_h and F_v multiplied by the arm length $l_h(\alpha_v)$ and l_v are equal to the torque's acting on the arm.

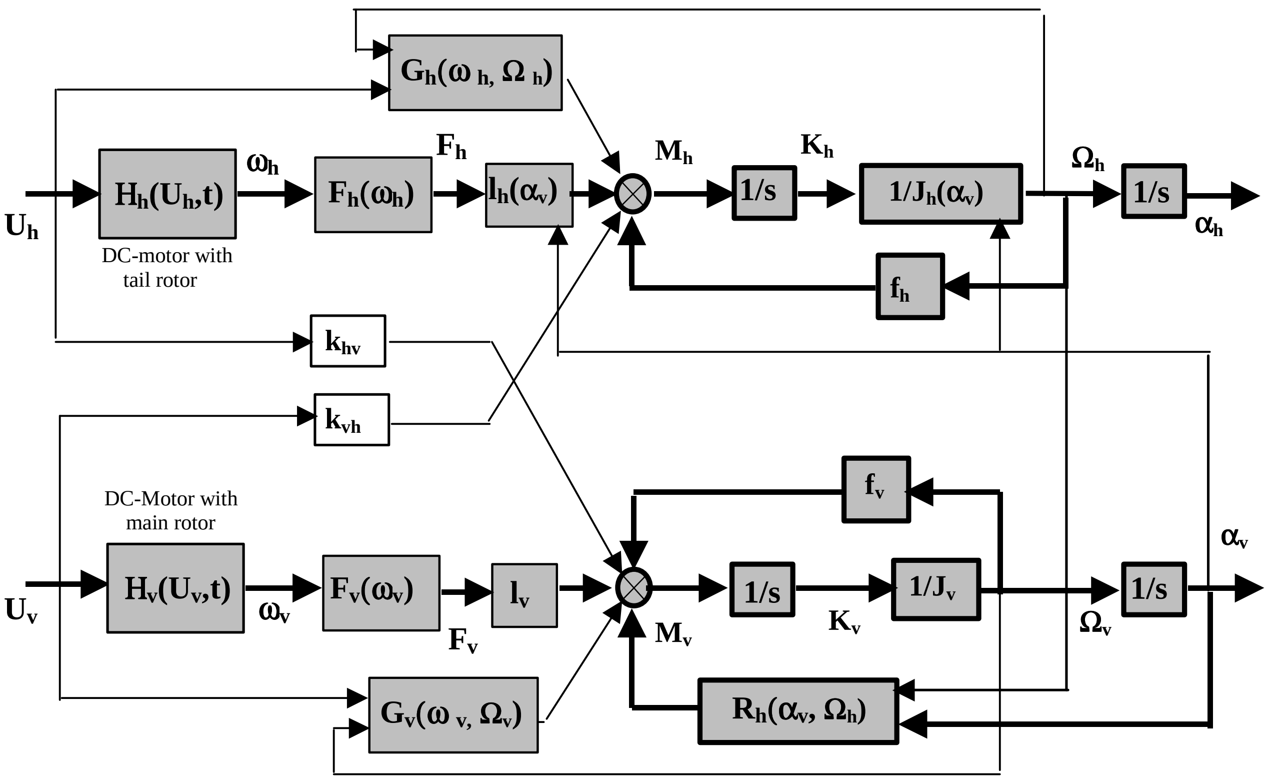

Fig. 1.1 Block diagram of the TRAS model

The following notation is used :

- α_h is horizontal position (azimuth position) of TRAS beam [rad];
- Ω_h is angular velocity (azimuth velocity) of TRAS beam [rad/s];
- U_h is horizontal DC-motor PWM voltage control input [ ];
- ω_h is rotational speed of tail rotor [rad/s] - non-linear function $\omega_h = H_h(U_h, t)$ [rad/s] ;
- F_h is aerodynamic force from tail rotor - non-linear function $F_h = F_h(\omega_h)$ [N];
- l_h is effective arm of aerodynamic force from tail rotor $l_h = l_h(\alpha_v)$ [m];
- J_h is non-linear function of moment of inertia with respect to vertical axis, $J_h = J_h(\alpha_v)$ [kg m^2];

M_h is horizontal turning torque [Nm];
K_h is horizontal angular momentum [N m s];
f_h is moment of friction force in vertical axis [N m];
α_v is vertical position (pitch position) of TRAS beam [rad];
Ω_v is angular velocity (pitch velocity) of TRAS beam [rad/s];
U_v is vertical DC-motor PWM voltage control input [ ];
ω_v is rotational speed of main rotor - non-linear function $\omega_v = H_v(U_v, t)$ [rad/s];
F_v is aerodynamic force from main rotor - non-linear function $F_v = F_v(\omega_v)$ [N];
l_v is arm of aerodynamic force from main rotor [m];
J_v is moment of inertia with respect to horizontal axis [kg m^2];
M_v is vertical turning moment [Nm];
K_v is vertical angular momentum [N m s];
f_v is moment of friction force in horizontal axis [N m];
R_h is vertical returning moment $R_h = f_{cf} + f_g = R_h(\alpha_v, \Omega_h)$ [N m];
J_{hv} is vertical angular momentum from tail rotor [N m s];
J_{vh} is horizontal angular momentum from main rotor [N m s];
t is time [s];
$1/s$ transfer function of an integrator
.

Controlling the system consists in stabilising the TRAS beam in an arbitrary, within practical limits, desired position (pith and azimuth) or making it track a desired trajectory. Both goals may be achieved by means of appropriately chosen controllers. The user can select between two types of PID controllers and state feedback (SF or LQ) controller.

Fig. 1.2 shows an aero-dynamical system similar to a helicopter. At both ends of a beam, joined to its base with an articulation, there are two propellers driven by DC-motors. The articulated joint allows the beam to rotate in such a way that its ends move on spherical surfaces. There is a counter-weight fixed to the beam and it determines a stable equilibrium position. The system is balanced in such a way, that when the motors are switched off, the main rotor end of beam is lowered. The controls of the system are the motor supply voltages.

The measured signals are: position of the beam in the space that is two position angles and angular velocities of the rotors. Angular velocities of the beam are software reconstructed by differentiating and filtering measured position angles of the beam.

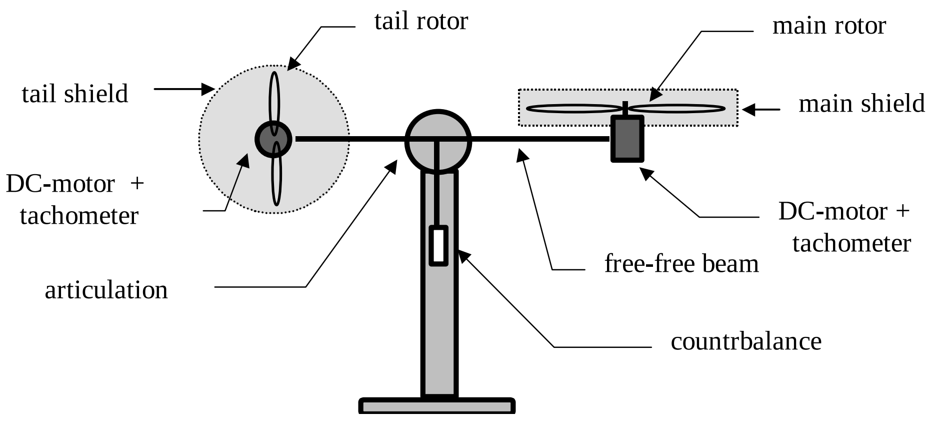

Fig. 1.2. Aero-dynamical model of TRMS

1.1 Non-linear model

The mathematical model is developed with some simplifying assumptions. First, it is assumed that the dynamics of the propeller subsystem can be described by first order differential equations. Further, it is assumed that friction in the system is of the viscous type. It is assumed also that the propeller-air subsystem could be described in accord with postulates of the flow theory.

The above assumptions allow us to define the problem clearly. First, consider the rotation of the beam in the vertical plane i.e. around the **horizontal axis**. Having in mind that the driving torque's are produced by the propellers the rotation can be described in principle as the motion of a pendulum. From the second dynamics law of Newton we obtain:

$$M_v = J_v \frac{d^2\alpha_v}{dt^2} \qquad (1)$$

where: M_v - is total moment of forces in the vertical plane,

J_v - is the sum of moments of inertia relative to the horizontal axis,

α_v - is the pitch angle of the beam.

Then:

$$M_v = \sum_{i=1}^{6} M_{vi}, \qquad J_v = \sum_{i=1}^{8} J_{vi}$$

To determine the moments of forces applied to the beam and making it rotate around the horizontal axis consider the situation shown in Fig. 1.3.

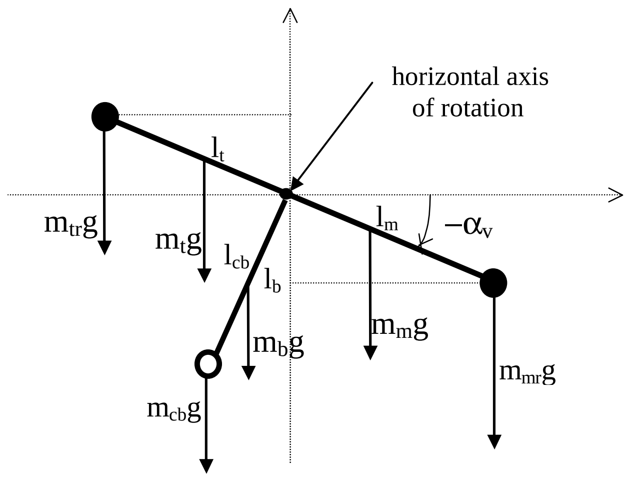

Fig. 1.3 Gravity forces in TRMS corresponding to the return torque which determines the equilibrium position of the system.

$$M_{v1} = g\left\{\left[\left(\frac{m_t}{2} + m_{tr} + m_{ts}\right)l_t - \left(\frac{m_m}{2} + m_{mr} + m_{ms}\right)l_m\right]\cos\alpha_v - \left[\frac{m_b}{2}l_b + m_{cd}l_{cb}\right]\sin\alpha_v\right\}$$

$$M_{v1} = g[(A - B)\cos\alpha_v - C\sin\alpha_v]$$

where:

$$A = \left(\frac{m_t}{2} + m_{tr} + m_{ts}\right)l_t$$

$$B = \left(\frac{m_m}{2} + m_{mr} + m_{ms} \right) l_m$$

$$C = \left[\frac{m_b}{2} l_b + m_{cd} l_{cb} \right]$$

where: M_{v1} - is the return torque corresponding to the forces of gravity,
m_{mr} - is the mass of the main DC-motor with main rotor,
m_m - is the mass of main part of the beam,
m_{tr} - is the mass of the tail motor with tail rotor,
m_t - is the mass of the tail part of the beam,
m_{cb} - is the mass of the counter-weight,
m_b - is the mass of the counter-weight beam,
m_{ms} - is the mass of the main shield,
m_{ts} - is the mass of the tail shield,
l_m - is the length of main part of the beam,
l_t - is the length of tail part of the beam,
l_b - is the length of the counter-weight beam,
l_{cb} - is the distance between the counter-weight and the joint.
g - is gravitational acceleration,

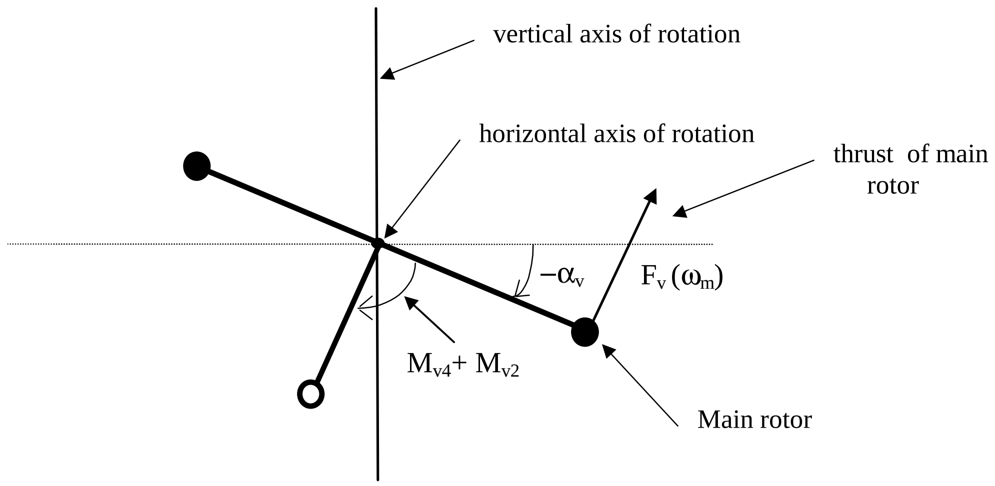

Fig. 1.4 Propulsive force moment and friction moment in TRMS

$$M_{v2} = l_m F_v(\omega_m)$$

M_{v2} - is the moment of the propulsive force produced by the main rotor,
ω_v - is angular velocity of the main rotor,
$F_v(\omega_v)$ - denotes the dependence of the propulsive force on the angular velocity of the rotor; it should be measured experimentally (see section 3.2.1).

$$M_{v3} = -\Omega_h^2 \left[\left(\frac{m_m}{2} + m_{mr} + m_{ms} \right) l_m + \left(\frac{m_t}{2} + m_{tr} + m_{ts} \right) l_t + m_{cb} l_{cb} + \frac{m_b}{2} l_b \right] \sin\alpha_v \cos\alpha_v$$

or in the compact form:

$$M_{v3} = -\Omega_h^2 (A + B + C)\sin\alpha_v \cos\alpha_v$$

M_{v3} - is the moment of centrifugal forces corresponding to the motion of the beam around the vertical axis,

and: $\Omega_h = \frac{d\alpha_h}{dt}$ (2)

Ω_h - is the angular velocity of the beam around the vertical axis, α_h - is the azimuth angle of the beam.

$$M_{v4} = -\Omega_v k_v$$

M_{v4} - is the moment of friction depending on the angular velocity of the beam around the horizontal axis.

where: $\Omega_v = \frac{d\alpha_v}{dt}$ (3)

Ω_v - is the angular velocity around the horizontal axis,

k_v - is a constant

M_{v5} - is the cross moment from U_h

$M_{v5} = U_h k_{hv}$

k_{hv} -is constant

M_{v6} - is the dumping tourque from rotatin propeller

$M_{v6} = -a_1 \Omega_v abs(\omega_v)$

a_1 - is constatnmt

According to Fig. 1.4 we can determine components of the moment of inertia relative to the horizontal axis. Notice that this moment is independent of the position of the beam.

$J_{v1} = m_{mr} l_m^2$,

$J_{v2} = m_m \frac{l_m^2}{3}$,

$J_{v3} = m_{cb} l_{cb}^2$

$J_{v4} = m_b \frac{l_b^2}{3}$,

$J_{v5} = m_{tr} l_t^2$,

$J_{v6} = m_t \frac{l_t^2}{3}$

$J_{v7} = \frac{m_{ms}}{2} r_{ms}^2 + m_{ms} l_m^2$,

$J_{v8} = m_{ts} r_{ts}^2 + m_{ts} l_t^2$

r_{ms} - is radius of main shield,

r_{ts} - is radius of tail shield

Similarly we can describe the motion of the beam around the **vertical axis**. Having in mind that the driving torque's are produced by the rotors and that the moment of inertia depends on the pitch angle of the beam the horizontal motion of the beam (around the vertical axis) can be described in principle as rotative motion of a solid:

$$M_h = J_h \frac{d^2\alpha_h}{dt^2} \qquad (4)$$

where: M_h - is the sum of moments of forces acting in the horizontal plane,

J_h - is the sum of moments of inertia relative to the vertical axis.

Then: $M_h = \sum_{i=1}^{4} M_{hi}$, $J_h = \sum_{i=1}^{8} J_{hi}$

To determine the moments of forces applied to the beam and making it rotate around the vertical axis consider the situation shown in Fig. 1.5.

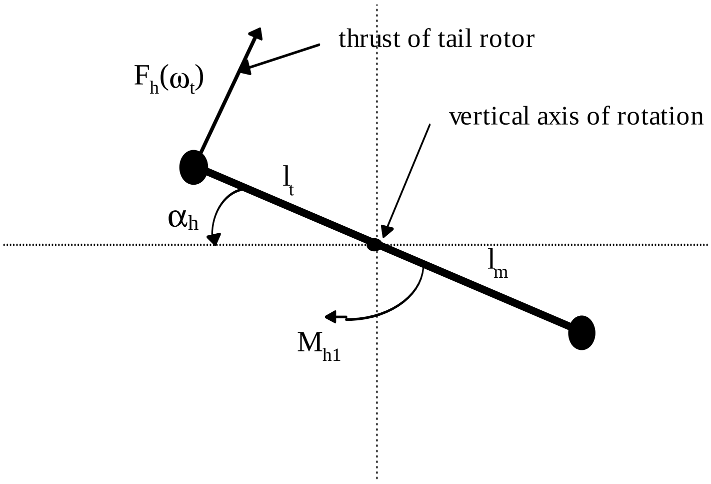

Fig. 1.5 Moments of forces in horizontal plane (as seen from above)

$$M_{h1} = l_t\, F_h(\omega_t) \cos\alpha_v$$

ω_t - is rotative velocity of tail rotor,

$F_h(\omega_t)$ - denotes the dependence of the propulsive force on the angular velocity of the tail rotor (should be determine experimentally, see section 3.2.2)

$$M_{h2} = -\Omega_h k_h$$

M_{h2} - is the moment of friction depending on the angular velocity of the beam around the vertical axis,

k_h - is constant.

M_{h3} - is the cross moment from U_v

$M_{h3} = U_v k_{vh}$

k_{vh} - is constant

M_{h4} - is the dumping tourque from rotating propeller

$M_{h4} = -a_2 \Omega_h abs(\omega_h)$

a_2 - is constatnt

According to Fig. 1.4 we can determine components of the moment of inertia relative to vertical axis (it depends on pitch position of the beam).

$$J_{h1} = \frac{m_m}{3}(l_m \cos\alpha_v)^2,$$
$$J_{h2} = \frac{m_t}{3}(l_t \cos\alpha_v)^2,$$
$$J_{h3} = \frac{m_b}{3}(l_b \sin\alpha_v)^2$$
$$J_{h4} = m_{tr}(l_t \cos\alpha_v)^2,$$
$$J_{h5} = m_{mr}(l_m \cos\alpha_v)^2,$$
$$J_{h6} = m_{cb}(l_{cb} \sin\alpha_v)^2$$
$$J_{h7} = \frac{m_{ts}}{2} r_{ts}^2 + m_{ts}(l_t \cos\alpha_v)^2,$$
$$J_{h8} = m_{ms} r_{ms}^2 + m_{ms}(l_m \cos\alpha_v)^2$$

or in the compact form:

$$J_h = D\cos^2\alpha_v + E\sin^2\alpha_v + F$$

where: D,E,F are constants:

$$D = \frac{m_b}{3} l_b^2 + m_{cb} l_{cb}^2,$$
$$E = \left(\frac{m_m}{3} + m_{mr} + m_{ms}\right) l_m^2 + \left(\frac{m_t}{3} + m_{tr} + m_{ts}\right) l_t^2,$$
$$F = m_{ms} r_{ms}^2 + \frac{m_{ts}}{2} r_{ts}^2$$

$$M_{v1} = g\{(A - B)\cos\alpha_v - C\sin\alpha_v\}$$

Using (1)-(4) we can write the equations describing the motion of the system as follows:

$$\frac{d\Omega_v}{dt} = \frac{l_m F_v(\omega_m) - \Omega_v k_v + g((A-B)\cos\alpha_v - C\sin\alpha_v) - \frac{1}{2}\Omega_h^2 (A+B+C)\sin 2\alpha_v U_h k_{hv} + U_h k_{hv} - a_1 \Omega_v abs(\omega_v)}{J_v} \quad (5)$$

$$\frac{d\alpha_v}{dt} = \Omega_v, \quad (6)$$

$$\frac{dK_h}{dt} = \frac{M_h}{J_h} = \frac{l_t F_h(\omega_t)\cos\alpha_v - \Omega_h k_h + U_v k_{vh} - a_2 \Omega_h abs(\omega_h)}{D\sin^2\alpha_v + E\cos^2\alpha_v + F} \quad (7)$$

$$\frac{d\alpha_h}{dt} = \Omega_h, \qquad \Omega_h = \frac{K_h}{J_h(\alpha_v)} \quad (8)$$

and two ecqations describing th motion of rotors:

$$I_h \frac{d\omega_h}{dt} = U_h - H_h^{-1}(\omega_h)$$

$$I_v \frac{d\omega_v}{dt} = U_v - H_v^{-1}(\omega_v)$$

I_h - moment of inertia of tail rotor

I_v - moment of inertia of main rotor.

The above model of the motor-propeller dynamics is obtained by substituting the non-linear system by a serial connection of a linear dynamic system and static non-linearity.

1.2 State equations

Finally, the mathematical model of the TRMS becomes as the set of four non-linear differential equations with two linear differential equations and four non-linear functions.

$U = \begin{bmatrix} u_h \\ u_v \end{bmatrix}$ is the input, $X = \begin{bmatrix} K_h \\ \alpha_h \\ \omega_h \\ K_v \\ \alpha_v \\ \omega_v \end{bmatrix}$ is the state, and $Y = \begin{bmatrix} \Omega_h \\ \alpha_h \\ \omega_t \\ \Omega_v \\ \alpha_v \\ \omega_v \end{bmatrix}$ is the output vector.

In order to apply the model for control of TRMS the parameters and non-linear functions should be determined first. They can be divided in to three groups:

- physical parameters
- non-linear static characteristics
- time constants of the linear part of the model

It is described in detail in the next section.
asic parameters and characteristics of the TRMS

1.3 Physical parameters

To obtain the values of model coefficients it is necessary to make some measurements. First, geometrical dimensions and moving masses of TRMS should be measured. Following are the results of measurements for a given laboratory set-up. The notations are given in Fig.1.3, Fig.1.4 and Fig.1.5.

$$m_{tr} = 0.154 \quad [kg]$$

$l_t = 0.216 \quad [m]$	$m_{mr} = 0.199 \quad [kg]$	
$l_m = 0.202 \quad [m]$	$m_{cb} = 0.024 \quad [kg]$	
$l_b = 0.15 \quad [m]$	$m_t = 0.031 \quad [kg]$	WAGI:
$l_{cb} = 0.15 \quad [m]$	$m_m = 0.029 \quad [kg]$	
$r_{ms} = 0.145 \quad [m]$	$m_b = 0.011 \quad [kg]$	
$r_{ts} = 0.10 \quad [m]$	$m_{ts} = 0.061 \quad [kg]$	
	$m_{ms} = 0.083 \quad [kg]$	

Using the above measurements the moment of inertia about the horizontal axis can be calculated as:

$$J_v = \sum_i^8 J_{iv} = 0.02421 [kg\, m^2]$$

The terms of the sum are calculated from elementary physics laws:

$$J_{v1} = m_{tr} l_t^2 = 0.00718 [kg\, m^2]$$
$$J_{v2} = m_{cb} l_{cb}^2 = 0.00054 [kg\, m^2]$$
$$J_{v3} = m_{mr} l_m^2 = 0.00811 [kg\, m^2]$$
$$J_{v4} = m_t l_t^2 / 3 = 0.00049 [kg\, m^2]$$
$$J_{v5} = m_m l_m^2 / 3 = 0.00040 [kg\, m^2]$$
$$J_{v6} = m_b l_b^2 / 3 = 0.00008 [kg\, m^2]$$
$$J_{v7} = m_{ms} (r_{ms}^2 / 2 + l_m^2) = 0.00426 [kg\, m^2]$$
$$J_{v8} = m_{ts} (r_{ts}^2 / 2 + l_t^2) = 0.00315 [kg\, m^2]$$

The calculated moment of inertia about the vertical axis is:

$$J_h = \sum_i^9 J_{hi}$$

where the terms of the sum are:

$$J_{h1} = m_t (l_t \cos\alpha_v)^2/3 \quad = 0.000482 \cos^2\alpha_v \quad [\mathrm{kg\,m^2}]$$
$$J_{h2} = m_m (l_m \cos\alpha_v)^2/3 \quad = 0.000394 \cos^2\alpha_v \quad [\mathrm{kg\,m^2}]$$
$$J_{h3} = m_b (l_b \sin a_v)^2/3 \quad = 0.000082 \sin^2\alpha_v \quad [\mathrm{kg\,m^2}]$$
$$J_{h4} = m_{mr} (l_m \cos\alpha_v)^2 \quad = 0.008119 \cos^2\alpha_v \quad [\mathrm{kg\,m^2}]$$
$$J_{h5} = m_{tr} (l_t \cos\alpha_v)^2 \quad = 0.007185 \cos^2\alpha_v \quad [\mathrm{kg\,m^2}]$$
$$J_{h6} = m_{cb} (l_{cb} \sin\alpha_v)^2 \quad = 0.00054 \quad \sin^2\alpha_v \quad [\mathrm{kg\,m^2}]$$
$$J_{h7} = 0.00063 \quad [\mathrm{kg\,m^2}]$$
$$J_{h8} = m_{ts} \left(r_{ts}^2/2 + l_t^2 \cos^2\alpha_v\right) \quad = 0.00031 + 0.00284 \cos^2\alpha_v \quad [\mathrm{kg\,m^2}]$$
$$J_{h9} = m_{ms} \left(r_{ms}^2/2 + l_m^2 \cos^2\alpha_v\right) = 0.00087 + 0.00387 \cos^2\alpha_v \quad [\mathrm{kg\,m^2}]$$

giving finally (Fig. 1.6):

$$J_h = \sum_i^9 J_{hi} = D\cos^2\alpha_v + E\sin^2\alpha_v + F = 0.022893 \cos^2\alpha_v + 0.0006225 \sin^2\alpha_v + 0.001267$$

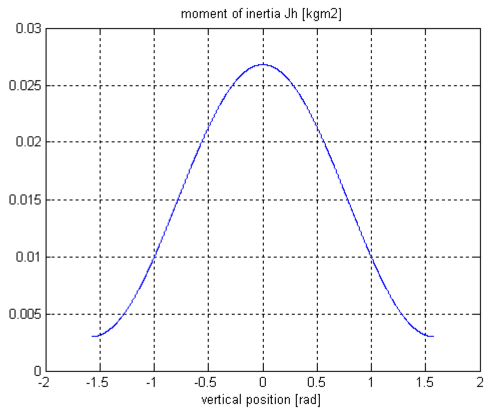

Fig. 1.6 Moment of inertia J_h

The returning torque from gravity forces is expressed by :

$$M_r = \sum_i^8 M_{ri}$$

and its components are given by:

$$M_{r1} = -9.81\, m_m l_m \cos\alpha_v /2 = -0.0287 \cos\alpha_v \text{ [N m]}$$
$$M_{r2} = -9.81\, m_{mr} l_m \cos\alpha_v = -0.3943 \cos\alpha_v \text{ [N m]}$$
$$M_{r3} = -9.81\, m_{ms} l_m \cos\alpha_v = -0.16 \cos\alpha_v \text{ [N m]}$$
$$M_{r4} = +9.81\, m_t l_t \cos\alpha_v /2 = 0.0328 \cos\alpha_v \text{ [N m]}$$
$$M_{r5} = +9.81\, m_{tr} l_t \cos\alpha_v = 0.3263 \cos\alpha_v \text{ [N m]}$$
$$M_{r6} = +9.81\, m_{ts} l_t \cos\alpha_v = 0.12925 \cos\alpha_v \text{ [N m]}$$
$$M_{r7} = -9.81\, m_b l_b \sin\alpha_v /2 = 0.01618 \sin\alpha_v \text{ [N m]}$$
$$M_{r8} = -9.81\, m_{cb} l_{cb} \sin\alpha_v = 0.03531 \sin\alpha_v \text{ [N m]}$$

giving finally (Fig.1.7):

$$M_r = \sum_i^8 M_{ri} = (-0.0947 \cos\alpha_v + 0.05149 \sin\alpha_v) \text{ [N m]}$$

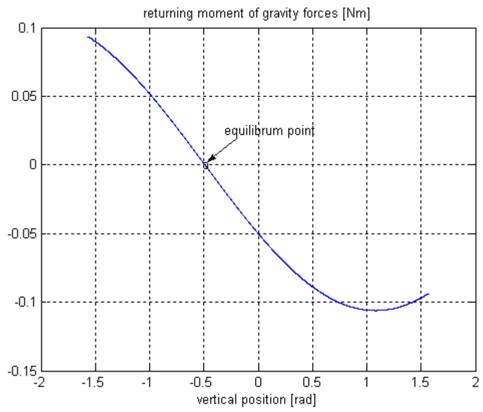

Fig. 1.7 Returning moment of gravity forces M_r

The moment of centrifugal forces is:

$$M_{cf} = \sum_i^6 M_{cfi}$$

Where:

$$M_{cf1} = (m_{tr} + m_{ts}) l_t^2\, \Omega_h^2 \cos\alpha_v \sin\alpha_v = 0.01003\, \Omega_h^2 \cos\alpha_v \sin\alpha_v \text{ [N m]}$$
$$M_{cf2} = m_t l_t^2\, \Omega_h^2 \cos\alpha_v \sin\alpha_v /4 = 0.00144\, \Omega_h^2 \cos\alpha_v \sin\alpha_v \text{ [N m]}$$
$$M_{cf3} = m_b l_b^2\, \Omega_h^2 \cos\alpha_v \sin\alpha_v /4 = 0.00025\, \Omega_h^2 \cos\alpha_v \sin\alpha_v \text{ [N m]}$$
$$M_{cf4} = m_{cb} l_{cb}^2\, \Omega_h^2 \cos\alpha_v \sin\alpha_v = 0.00054\, \Omega_h^2 \cos\alpha_v \sin\alpha_v \text{ [N m]}$$
$$M_{cf5} = m_m l_m^2\, \Omega_h^2 \cos\alpha_v \sin\alpha_v /4 = 0.00118\, \Omega_h^2 \cos\alpha_v \sin\alpha_v \text{ [N m]}$$
$$M_{cf6} = (m_{mr} + m_{ms}) l_m^2\, \Omega_h^2 \cos\alpha_v \sin\alpha_v = 0.01150\, \Omega_h^2 \cos\alpha_v \sin\alpha_v \text{ [N m]}$$

giving finally (Fig.1.8):

$$M_{cf} = \sum_{i}^{6} M_{cfi} = 0.024946 \;\; \Omega_h^2 \cos\alpha_v \; \sin\alpha_v \;\; [\mathrm{N\,m}]$$

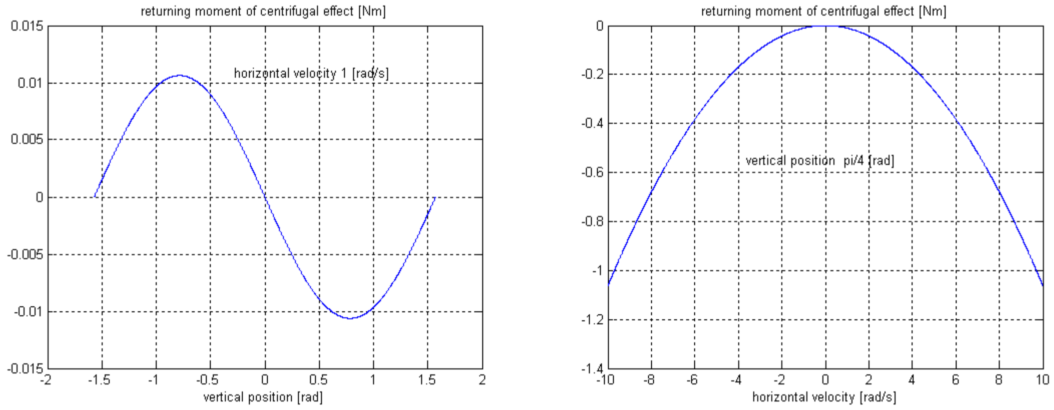

Fig. 1.8 Centrifugal moment for Ω_h=1 [rad/s]

1.4 Static characteristics

It is necessary to identify the following non-linear functions:

- Two non-linear input characteristics determining dependence of DC- motor rotational speed on input voltage:

$$\omega_v = H_v(U_v)\,,\;\; \omega_h = H_h(U_h)$$

- Two non-linear characteristics determining dependence of propeller thrust on DC- motor rotational speeds:

$$F_h = F_h(\omega_h)\;\;,\, F_v = F_v(\omega_v)$$

The static characteristics of the propellers should be measured in the case when not delivered with equipment documentation or if the propellers were changed by user. In this case a proper electronic balance with

1.4.1 Main rotor characteristics

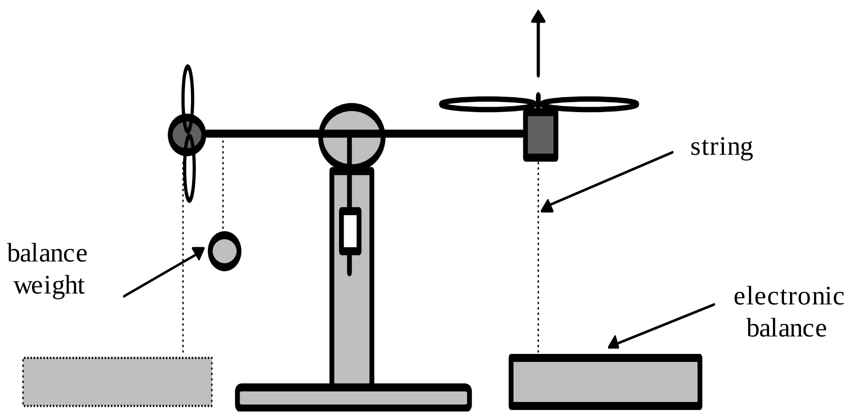

Fig. 1.9 Measuring of main rotor characteristics.

To make measurements correctly block the beam so that it could not rotate around the vertical axis, place the electronic balance under the beam in such a way that it is pulled by the propeller straight up. To balance the beam in the horizontal position attach a weight to the beam (as on Fig.1.9) Connect the voltage output of the electronic balance to A/D input of the RT-DAC-PCI data acquisition board.

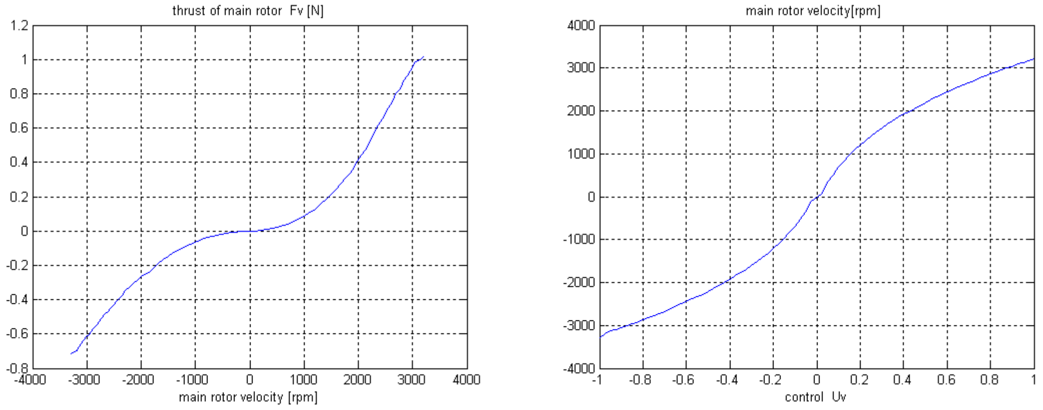

Fig. 1.10 Measured characteristics of the main rotor.

For further applications the measured characteristics should be substituted by their polynomial approximations . For this purposes one can use the MATLAB ***polyfit.m*** function.

1.4.2 Tail rotor characteristics

Fig. 1.12 shows laboratory set-up for measuring thrust and rotational speed of the tail rotor.

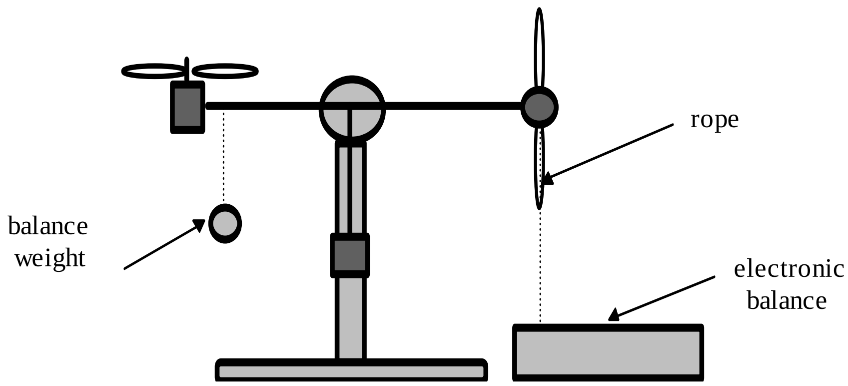

Fig. 1.11 Laboratory set-up for tail rotor characteristics

To measure the characteristic execute the following routine:
The above routine sends a normalised input signal (from **0** to **-1**) which corresponds the 0 to-5[V] on the output of PCL-812 card to the tail DC-motor and reads tail rotor rotational speed and thrust. Than to measure the characteristics for input range from **0** to **+1** change the balance position as in Fig.3.7 and execute the routine having previously changed the 12^{th} line to:

The obtained measurements should be translated from internal system units to physical units (newtons [N]) taking into account construction of the balance.
Example characteristics are given in Fig.3.8.

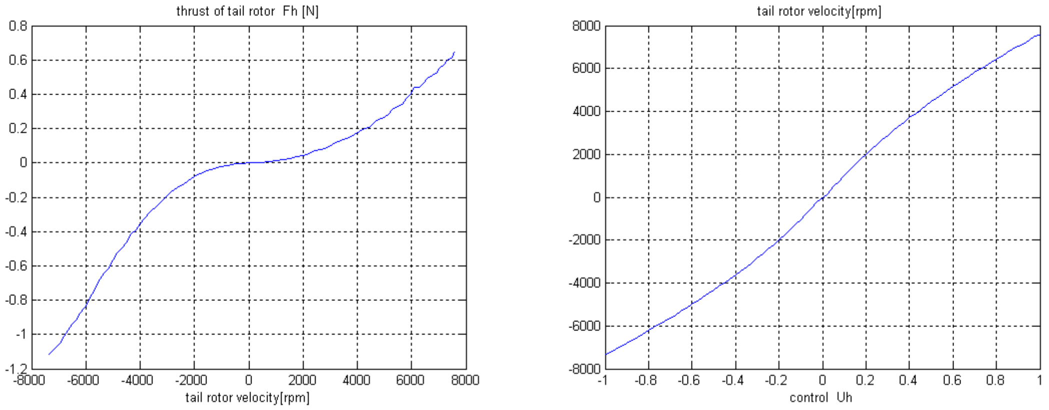

Fig. 1.12 Characteristics measured for the tail rotor.